Double Integrals in Polar Form

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Overview

Integrals are sometimes easier to evaluate if we change to polar coordinates.

We discuss in the lecture how to accomplish the change and how to evaluate integrals over regions whose boundaries are given by polar equations.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O. Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.
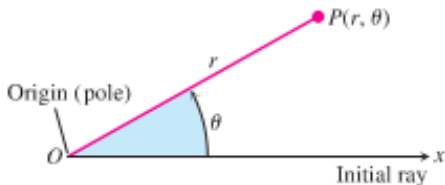
To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

Polar coordinates are not unique

$$P(r, \theta)$$

Directed distance
from O to P

Directed angle from
initial ray to OP

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique.

While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates.
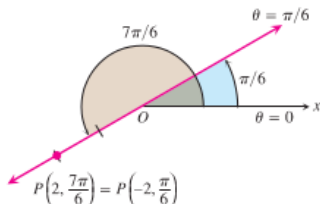
Polar coordinates are not unique : Example
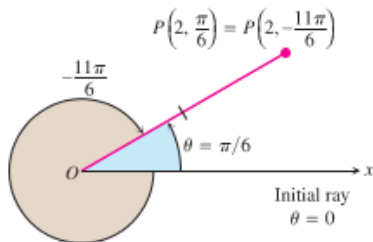
The point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2, \theta = \pi/6$.

It also has coordinates $r = 2, \theta = -11\pi/6$.

Polar coordinates can be negative *r*-values : Example

$$P\left(2, \frac{7\pi}{6}\right) = P\left(-2, \frac{\pi}{6}\right)$$

In some situations we allow *r* to be negative. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians counterclockwise from the initial ray and going forward 2 units. It can also be reached by turning $\pi/6$ radians counterclockwise from the initial ray going **backward** 2 units. So the point also has polar coordinates $r = 2, \theta = \pi/6$.

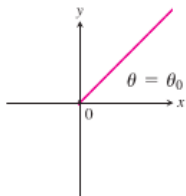# Polar Equations and Graphs

The equation

$$r = a$$

represents circle of radius $|a|$ centered at O.
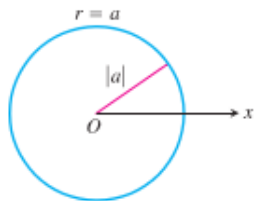
The equation

$$\theta = \theta_0$$

represents line through O making an angle θ with the initial ray.

Equatins Relating Polar and Cartesian Coordinates
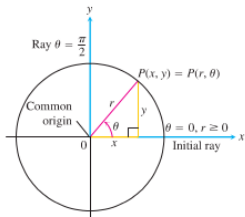
The Cartesian and polar coordinate systems are related by the following equations.

$$
\begin{aligned}
x &= r \cos \theta \\
y &= r \sin \theta \\
r^2 &= x^2 + y^2 \\
\tan \theta &= \frac{y}{x}.
\end{aligned}
$$

The first two of these equations uniquely determine the Cartesian coordinates x and y given the polar coordinates r and θ. On the other hand, if x and y are given, the third equation gives two possible choices for r (a positive and a negative value). For each $(x, y) \neq (0, 0)$, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (x, y).

Equivalent Expressions

The following are some equivalent equations expressed in terms of both polar coordinates and Cartesian coordinates

Polar Equation	Cartesian Equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Graphing in Polar Coordinates : Symmetry Tests for Polar Graphs

We now discuss techniques for graphing equations in polar coordinates using symmetries.

Symmetry about the x-axis:

If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi, -\theta)$ lies on the graph.
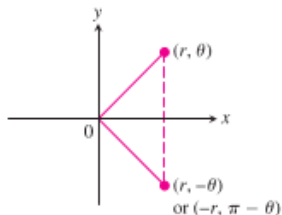
Graphing in Polar Coordinates : Symmetry Tests for Polar Graphs
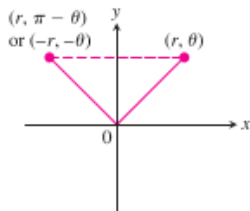
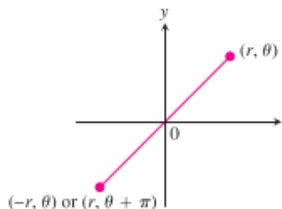**Symmetry about the y-axis:**

If the point (r, θ) lies on the graph, then the point $(r, \pi, -\theta)$ or $(-r, -\theta)$ lies on the graph.

Graphing in Polar Coordinates : Symmetry Tests for Polar Graphs

Symmetry about the origin:

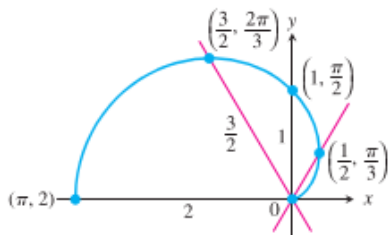If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph.

Graph the curve $r = 1 - \cos\theta$

θ	$r = 1 - \cos\theta$
0	0
$\dfrac{\pi}{3}$	$\dfrac{1}{2}$
$\dfrac{\pi}{2}$	1
$\dfrac{2\pi}{3}$	$\dfrac{3}{2}$
π	2

Graph the curve $r = 1 - \cos \theta$
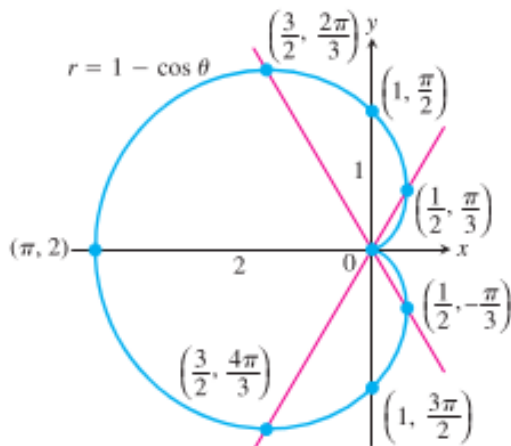
Graph the curve $r^2 = 4\cos\theta$

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

$r^2 = 4\cos\theta$

Loop for $r = -2\sqrt{\cos\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Loop for $r = 2\sqrt{\cos\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
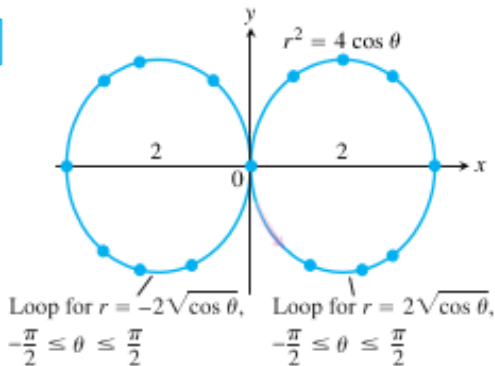
A Technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ)-values, plot the corresponding points, and connect them in order of increasing θ. This can work well if enough points have been plotted to reveal all the loops and dimples in the graph.

Another method of graphing that is usually quicker and more reliable is to

(a) first graph $r = f(\theta)$ in the **Cartesian $r\theta$-plane**,
(b) then use the Cartesian graph as a "table" and guide to sketch the **polar coordinate graph**.

This method is better than simple point plotting because the first Cartesian graph, even when hastily drawn, shows at a glance where r is positive, negative, and nonexistent, as well as where r is increasing and decreasing.

Graph the curve $r^2 = \sin 2\theta$

We begin by plotting r^2 (not r) as a function of θ in the Cartesian $r^2\theta$-plane.
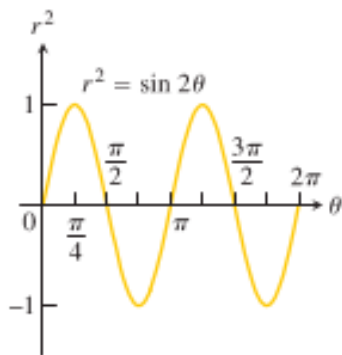
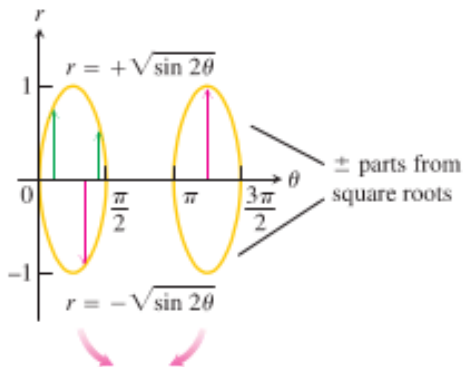# Graph the curve $r^2 = \sin 2\theta$

We began by plotting r^2 (not r) as a function of θ in the Cartesian $r^2\theta$-plane. We pass from there to the graph of $r = \pm\sqrt{\sin 2\theta}$ in the $r\theta$-plane, and then draw the polar graph.

Graph the curve $r^2 = \sin 2\theta$

The graph covers the final polar graph twice. We could have managed with either loop alone, with the two upper halves, or with the two lower halves.

The double covering does no harm, however, and we actually learn a litter more about the behaviour of the function this way.
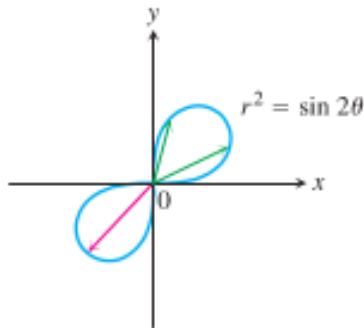
Integrals in Polar Coordinates

When we defined the double integral of a function over a region R in the xy-plane, we began by cutting R into rectangles whose sides were parallel to the coordinate axes.

These were the natural shapes to use because their sides have either constant x-values or constant y-values.

In polar coordinates, the natural shape is a "polar rectangle" whose sides have constant r- and θ-values.

Integrals in Polar Coordinates

Suppose that a function

$$f(r, \theta)$$

is defined over a region R that is bounded by the rays

$$\theta = \alpha \quad \text{and} \quad \theta = \beta$$

and by the continuous curves

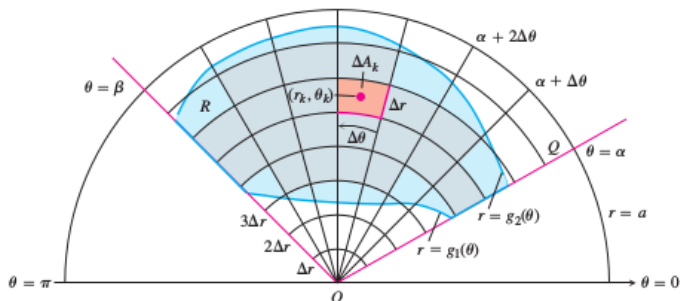$$r = g_1(\theta) \quad \text{and} \quad r = g_2(\theta).$$

Integrals in Polar Coordinates

Suppose also that

$$0 \le g_1(\theta) \le g_2(\theta) \le a$$

for every value of θ between α and β. Then R lies in a fan-shaped region Q defined by the inequalities

$$0 \le r \le a \quad \text{and} \quad \alpha \le \theta \le \beta.$$

Integrals in Polar Coordinates

We cover Q by a grid of circular arcs and rays.

The arcs are cut from circles centered at the origin, with radii

$$\Delta r, \ 2\Delta r, \ \ldots, \ m\Delta r,$$

where $\Delta r = a/m$.

The rays are given by

$$\theta = \alpha, \ \theta = \alpha + \Delta\theta, \ \theta = \alpha + 2\Delta\theta, \ \ldots, \ \theta = \alpha + m'\Delta\theta = \beta,$$

where $\Delta\theta = (\beta - \alpha)/m'$.

The arcs and rays partition Q into small patches called "polar rectangles."

Integrals in Polar Coordinates

We number the polar rectangles that lie inside R (the order does not matter), calling their areas

$$\Delta A_1, \Delta A_2, \ldots, \Delta A_n.$$

We let (r_k, θ_k) be any point in the polar rectangle whose area is ΔA_k.

We then form the sum

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k) \Delta A_k.$$

Integrals in Polar Coordinates

If f is continous throughout R, this sum will approach a limit as we refine the grid to make Δr and $\Delta \theta$ go to zero.

The limit is called the double integral of f over R. In symbols,

$$\lim_{n \to \infty} S_n = \iint\limits_R f(r, \theta) \; dA.$$

To evaluate this limit, we first have to write the sum S_n in a way that expresses ΔA_k in terms of Δr and $\Delta \theta$.

For convenience we choose r_k to be the average of the radii of the inner and outer acrs bounding the kth polar rectangle ΔA_k.

Integrals in Polar Coordinates

The radius of the inner arc bounding ΔA_k is then $r_k - (\Delta r/2)$.
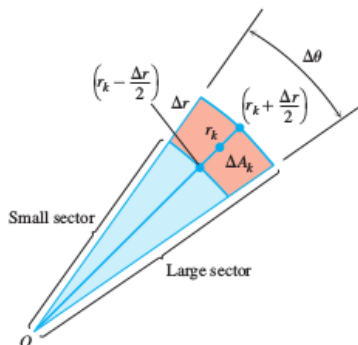
The radius of the outer arc is $r_k + (\Delta r/2)$.

Integrals in Polar Coordinates

The area of a wedge-shaped sector of a circle having radius r and angle θ is

$$A = \frac{1}{2}\theta \ r^2,$$

as can be seen by multiplying πr^2, the area of the circe, by $\theta/2\pi$, the fraction of the circle's area contained in the wedge.

So the areas of the circular sectors subtended by these arcs at the origin are

Inner radius : $\frac{1}{2}\left(r_k - \dfrac{\Delta r}{2}\right)^2 \Delta\theta.$

Outer radius : $\frac{1}{2}\left(r_k + \dfrac{\Delta r}{2}\right)^2 \Delta\theta.$

Integrals in Polar Coordinates

Therefore,

$$
\begin{aligned}
\Delta A_k &= \text{area of large sector } - \text{ area of small sector} \\
&= \frac{\Delta \theta}{2}\Big[\Big(r_k + \frac{\Delta r}{2}\Big)^2 - \Big(r_k - \frac{\Delta r}{2}\Big)^2\Big] \\
&= \frac{\Delta \theta}{2}(2r_k \,\Delta r) \\
&= r_k \,\Delta r \,\Delta \theta.
\end{aligned}
$$

Combining this result with the sum defining S_n gives

$$
S_n = \sum_{n=1}^{n} f(r_k, \theta_k) \; r_k \; \Delta r \; \Delta \theta.
$$

Fubini's Theorem

As $n \to \infty$ and the values of Δr and $\Delta \theta$ approach zero, these sums converge to the double integral

$$\lim_{n \to \infty} S_n = \iint\limits_R f(r, \theta) r \, dr \, d\theta.$$

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint\limits_R f(r, \theta) \, r \, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta) \, r \, dr \, d\theta.$$

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates.

Finding Limits of Integration

To evaluate $\iint\limits_R f(r, \theta) \, dA$ over a region R in polar coordinates,

integrating first with respect to r and then with respect to θ, take the following steps.

1. **Sketch:** Sketch the region and label the bounding curves.

2. **Find the r-limits of integration:** Imagine a ray L from the the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis.

3. **Find the θ-limits of integration:** Find the smallest and largest θ-values that bound R. These are the θ-limits of integration.

Finding Limits of Integration
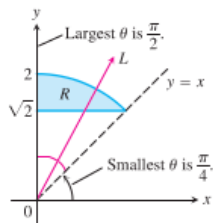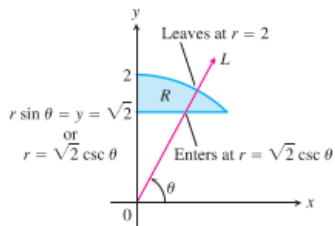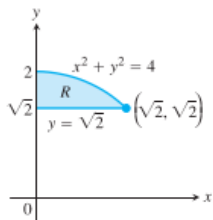
Example 1.

Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside cardioid

$$r = 1 + \cos\theta$$

and outside the circe $r = 1$.
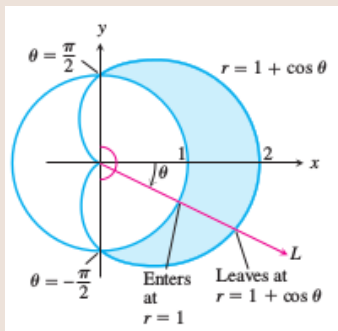
Area in Polar Coordinates

If $f(r, \theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R. The area of a closed and bounded region R in the polar coordinate plane is $A = \iint\limits_{R} r \, dr \, d\theta$.

Example 2.

Find the area enclosed by one leaf of curve given by $r^2 = 4 \cos 2\theta$.

Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint\limits_{R} f(x, y) \ dx \ dy$ into a polar integral has two steps. First substitute $x = r \cos \theta$ and $y = r \sin \theta$, and replace $dx \ dy$ by $r \ dr \ d\theta$ in the Cartesian integral. Then supply polar limits for the boundary of R. The Cartesian integral then becomes
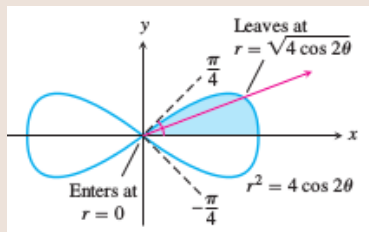
$$\iint\limits_{R} f(x, y) \ dx \ dy = \iint\limits_{G} f(r \cos \theta, r \sin \theta) \ r \ dr \ d\theta,$$

where G denotes the region of integration in polar coordinates. This is like the substitution method (to variables to substitute).

Notice that $dx \ dy$ is not replaced by $dr \ d\theta$ but by $r \ dr \ d\theta$.

Example 3.

Find the polar moment of inertia about the orgin of a thin place of density

$$\delta(x, y) = 1$$

bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.
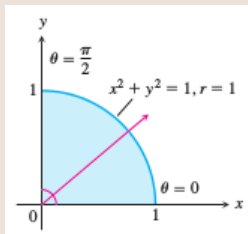
Example 4.

Evaluate $\iint\limits_{R} e^{x^2+y^2} \, dy \, dx$, where R is the semicircular region bounded by

the x-axis and the curve $y = \sqrt{1-x^2}$.
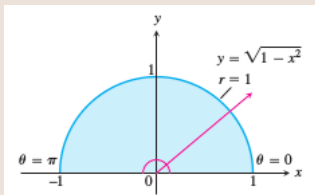
Example 5.

Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \, dx.$$

Solution : Integration with respect to y gives

$$\int_0^1 \left(x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} \right) \, dx,$$

an integral difficult to evaluate without tables.

Things go better if we change the original integral to polar coordinates. The region of integration in Cartesian coordinates is given by the inequalities $0 \leq y \leq \sqrt{1-x^2}$ and $0 \leq x \leq 1$, which correspond to the interior of the unit quarter circle $x^2 + y^2 = 1$ in the first quadrant.

Solution (contd...)
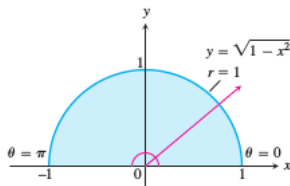
Substituting the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq 1$, and replacing $dx\,dy$ by $r\,dr\,d\theta$ in the double integral, we get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx = \int_0^{\pi/2} \int_0^1 (r^2)r\,dr\,d\theta = \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\,d\theta = \frac{\pi}{8}.$$

Why is the polar coordinate transformation so effective here? One reason is that $x^2 + y^2$ simplifies to r^2. Another is that the limits of integration become constants.

Example 6.

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

Solution :

The region of integration R is the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by $r = 1, 0 \leq \theta \leq 2\pi$.

The solid region is shown in the figure.
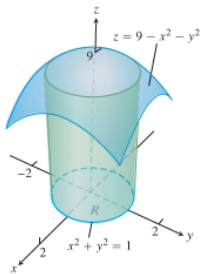
The volume is given by the double integral

$$\iint\limits_{R} (9 - x^2 - y^2) \, dA = \int_0^{2\pi} \int_0^1 (9 - r^2) r \, dr \, d\theta = \frac{17\pi}{2}.$$

Example 7.

Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 + 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution :
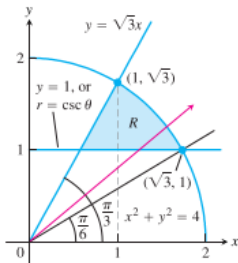
The area of the region R is

$$\iint\limits_R dA = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^2 r\, dr\, d\theta = \frac{\pi - \sqrt{3}}{3}.$$

Regions in Polar Coordinates

Exercise 8.

In the following exercises, describe the given region in polar coordinates.

1.

4.

2.

5.
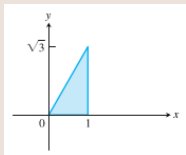
3. *The region enclosed by the circle $x^2 + y^2 = 2x$.*
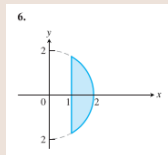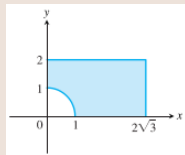
6. *The region enclosed by the semicircle $x^2 + y^2 = 2y, y \geq 0$.*

Solution for the Exercise 8

1. $x^2 + y^2 = 1^2 \Rightarrow r = 1, x^2 + y^2 = 4^2 \Rightarrow r = 4 \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 4$

2. $x = 1 \Rightarrow r = \sec\theta, y = \sqrt{3}x \Rightarrow \theta = \frac{\pi}{3} \Rightarrow 0 \leq \theta \leq \frac{\pi}{3}, 0 \leq r \leq \sec\theta$

3. $x^2 + y^2 = 2x \Rightarrow r = 2\cos\theta \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2\cos\theta$
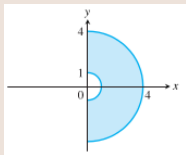
4. $x^2 + y^2 = 2^2 \Rightarrow r = 2, x = 1 \Rightarrow r = \sec\theta; 2 = \sec\theta \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3} \Rightarrow -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, \sec\theta \leq r \leq 2$

5. $x^2 + y^2 = 1^2 \Rightarrow r = 1, x = 2\sqrt{3} \Rightarrow r = 2\sqrt{3}\sec\theta, y = 2 \Rightarrow r = 2\csc\theta; 2\sqrt{3}\sec\theta = 2\csc\theta \Rightarrow \theta = \frac{\pi}{6} \Rightarrow 0 \leq \theta \leq \frac{\pi}{6}, 1 \leq r \leq 2\sqrt{3}\sec\theta, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 2\sqrt{3}\csc\theta$

6. $x^2 + y^2 = 2y \Rightarrow r = 2\sin\theta \Rightarrow 0 \leq \theta \leq \pi, 0 \leq r \leq 2\sin\theta$

Evaluating Polar Integrals

Exercise 9.

In the following exercises, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

1. $\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} dy \, dx$

2. $\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \, dx$

3. $\int_{0}^{2} \int_{0}^{x} y \, dy \, dx$

4. $\int_{1}^{\sqrt{3}} \int_{1}^{x} dy \, dx$

5. $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2} \, dy \, dx$

6. $\int_{0}^{\ln 2} \int_{0}^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} \, dx \, dy$

7. $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1) \, dx \, dy$

8. $\int_{1}^{2} \int_{0}^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2} \, dy \, dx$

1. $\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} dy\ dx = \int_{0}^{x} \int_{0}^{1} r\ dr\ d\theta = \frac{1}{2} \int_{0}^{x} d\theta = \frac{\pi}{2}$

2. $\int_{-a}^{a} \int_{\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\ dx = \int_{0}^{2x} \int_{0}^{a} r\ dr\ d\theta = \frac{a^2}{2} \int_{0}^{2x} d\theta = \pi a^2$

3. $\int_{0}^{2} \int_{0}^{x} = y\ dy\ dx = \int_{0}^{x/4} \int_{0}^{2\sec\theta} r^2\ \sin\theta\ dr\ d\theta = \frac{8}{3} \int_{0}^{x/4} \tan\theta \sec^2\theta\ d\theta = \frac{4}{3}$

4. $\int_{1}^{\sqrt{3}} \int_{1}^{x} dy\ dx = \int_{x/6}^{x/4} \int_{\csc\theta}^{\sqrt{3}\sec\theta} r\ dr\ d\theta = \int_{x/6}^{x/4} \left(\frac{3}{2}\sec^2\theta - \frac{1}{2}\csc^2\theta\right) d\theta = \left[\frac{3}{2}\tan\theta + \frac{1}{2}\cot\theta\right]_{x/6}^{x/4} = 2 - \sqrt{3}$

5. $\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)} dy\ dx = 4 \int_{0}^{x/2} \int_{0}^{1} \frac{2x}{(1+x^2)^2} dr\ d\theta = 4 \int_{0}^{x/2} \left[-\frac{1}{1+x^2}\right]_{0}^{1} d\theta = 2 \int_{0}^{x/2} d\theta = \pi$

6. $\int_{0}^{\ln 2} \int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}} dx\ dy = \int_{0}^{x/2} \int_{0}^{\ln 2} re^x\ dr\ d\theta = \int_{0}^{x/2} (2\ln 2 - 1) d\theta = \frac{\pi}{2}(2\ln 2 - 1)$

7. $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1) dx\ dy = 4 \int_{0}^{x/2} \int_{0}^{1} \ln(r^2+1) r\ dr\ d\theta = 2 \int_{0}^{x/2} (\ln 4 - 1) d\theta = \pi(\ln 4 - 1)$

8. $\int_{1}^{2} \int_{0}^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2} dy\ dx = \int_{0}^{\pi/4} \int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4} r\ dr\ d\theta = \int_{0}^{\pi/4} \left[-\frac{1}{2x^2}\right]_{\sec\theta}^{2\cos\theta} d\theta == \int_{0}^{\pi/4} \left(\frac{1}{2}\cos^2\theta - \frac{1}{8}\sec^2\theta\right) d\theta = \left[\frac{1}{4}\theta + \frac{1}{8}\sin 2\theta - \frac{1}{8}\tan\theta\right]_{0}^{\pi/4} = \frac{\pi}{16}$

Exercise 10.

In the following exercises, sketch the region of integration and convert each polar integral or sum of integrals to a Cartesian integral or sum of integrals. Do not evaluate the integrals.

1. $\displaystyle\int_0^{\pi/2} \int_0^1 r^3 \sin\theta \cos\theta \, dr \, d\theta$

2. $\displaystyle\int_0^{\pi/4} \int_0^{2\sec\theta} r^5 \sin^2\theta \, dr \, d\theta$

3. $\displaystyle\int_0^{\tan^{-1}\frac{4}{3}} \int_0^{3\sec\theta} r^7 \, dr \, d\theta + \int_{\tan^{-1}\frac{4}{3}}^{\pi/2} \int_0^{4\csc\theta} r^7 \, dr \, d\theta$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} x \, y \, dy \, dx$$

or

$$\int_0^1 \int_0^{\sqrt{1-y^2}} x \, y \, dx \, dy$$

$$\int_0^2 \int_0^x y^2(x^2 + y^2)dy \ dx$$

or

$$\int_0^2 \int_y^2 y^2(x^2 + y^2)dx \ dy$$
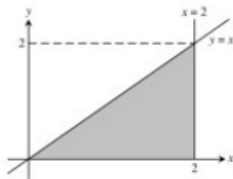
Solution for (3.) in Exercise 10

$$\int_0^3 \int_0^4 (x^2 + y^2)^3 dy \ dx$$

or

$$\int_0^4 \int_0^3 (x^2 + y^2)^3 dx \ dy$$
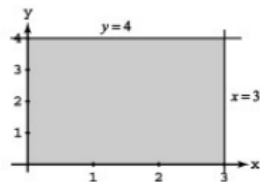
Exercise 11.

Set up (do not evaluate) polar integrals to find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$ in the following orders of integration.

(a) *dr dθ*   (b) *dθ dr.*

Solution for the Exercise 11
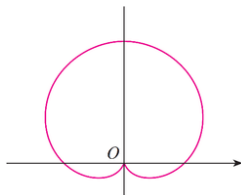
(a)
$$Area = \int_0^{\pi/2} \int_0^{1+\sin\theta} r \ dr \ d\theta.$$

(b)
$$Area = \int_0^1 \int_0^{\pi/2} r \ d\theta \ dr + \int_1^2 \int_{\sin^{-1}(r-1)}^{\pi/2} r \ d\theta \ dr.$$

Area in Polar Coordinates

Exercise 12.

1. *Find the area of the region cut from the first quadrant by the curve* $r = 2(2 - \sin 2\theta)^{1/2}$.

2. Cardioid overlapping a circle : *Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.*

3. One leaf of a rose : *Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.*

4. Snail shell : *Find the area of the region enclosed by the positive x-axis and spiral $r = 4\theta/3$, $0 \le \theta \le 2\pi$. The region looks like a snail shell.*

5. Cardioid in the first quadrant : *Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.*

6. Overlapping cardioids : *Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.*

Solution for the Exercise 12

1. $\displaystyle\int_0^{\pi/2}\int_0^{2\sqrt{2-\sin 2\theta}} r\ dr\ d\theta = 2\int_0^{\pi/2}(2-\sin 2\theta)d\theta = 2(\pi-1)$

2. $A = 2\displaystyle\int_0^{\pi/2}\int_1^{1+\cos\theta} r\ dr\ d\theta = \int_0^{\pi/2}(2\cos\theta+\cos^2\theta)d\theta = \dfrac{8+\pi}{4}$

3. $A = 2\displaystyle\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\ dr\ d\theta = 144\int_0^{\pi/6}\cos^2 3\theta\ d\theta = 12\pi$

4. $A = \displaystyle\int_0^{2\pi}\int_0^{4\theta/3} r\ dr\ d\theta = \dfrac{8}{9}\int_0^{2\pi}\theta^2\ d\theta = \dfrac{64\pi^4}{27}$

5. $A = \displaystyle\int_0^{\pi/2}\int_0^{1+\sin\theta} r\ dr\ d\theta = \dfrac{1}{2}\int_0^{\pi/2}\left(\dfrac{3}{2}+2\ \sin\theta-\dfrac{\cos 2\theta}{2}\right)d\theta = \dfrac{3\theta}{8}+1$

6. $A = 4\displaystyle\int_0^{\pi/2}\int_0^{1-\cos\theta} r\ dr\ d\theta = 2\int_0^{\pi/2}\left(\dfrac{3}{2}-2\cos\theta+\dfrac{\cos 2\theta}{2}\right)d\theta = \dfrac{3\pi}{2}-4$

Average values

Exercise 13.

In polar coordinates, the average value *of a function over a region R is given by* $\frac{1}{Area(R)} \iint\limits_{R} f(r, \theta) r \, dr \, d\theta$.

1. Average height of a hemisphere : *Find the average height of the hemispherical surface* $z = \sqrt{a^2 - x^2 - y^2}$ *above the disk* $x^2 + y^2 \leq a^2$ *in the xy-plane.*

2. Average height of a cone : *Find the average height of the (single) cone* $z = \sqrt{x^2 + y^2}$ *above the disk* $x^2 + y^2 \leq a^2$ *in the xy-plane.*

3. Average distance from interior of disk to center : *Find the average distance from a point* $P(x, y)$ *in the disk* $x^2 + y^2 \leq a^2$ *to the origin.*

4. Average distance squared from a point in a disk to a point in its boundary : *Find the average value of the square of the distance from the point* $P(x, y)$ *in the disk* $x^2 + y^2 \leq 1$ *to the boundary point* $A(1, 0)$.

Solution for the Exercise 13

1. average $= \dfrac{4}{\pi a^2} \displaystyle\int_0^{x/2} \int_0^a r\sqrt{a^2 - r^2}\, dr\ d\theta = \dfrac{4}{3\pi a^2} \int_0^{x/2} a^3 d\theta = \dfrac{2a}{3}$

2. average $= \dfrac{4}{\pi a^2} \displaystyle\int_0^{x/2} \int_0^a r^2 dr\ d\theta = \dfrac{4}{3\pi a^3} \int_0^{x/2} a^3 d\theta = \dfrac{2a}{3}$

3. average $= \dfrac{1}{\pi a^2} \displaystyle\int_{-a}^a \int_{\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2 + y^2}\, dy\ dx = \dfrac{1}{\pi a^2} \int_0^{2x} \int_0^a r^2 dr\ d\theta = \dfrac{a}{3\pi} \int_0^{2x} d\theta = \dfrac{2a}{3}$

4. average $= \dfrac{1}{\pi} \displaystyle\int_R \int [(1-x)^2 + y^2]\, dy\ dx = \dfrac{1}{\pi} \int_0^{2x} \int_0^1 [(1 - r\ \cos\theta)^2 + r^2\ \sin^2\theta]r\ dr\ d\theta =$

$\dfrac{1}{\pi} \displaystyle\int_0^{2x} \int_0^1 (r^3 - 2r^2 \cos\theta + r)\, dr\ d\theta = \dfrac{1}{\pi} \int_0^{2x} \left(\dfrac{3}{4} - \dfrac{2\cos\theta}{3} \right) d\theta = \dfrac{1}{\pi} \Big[\dfrac{3}{4}\theta - \dfrac{2\sin\theta}{3} \Big]_0^{2x} = \dfrac{3}{2}$

Exercises

Exercise 14.

1. Converting to a polar integral : Integrate
   $f(x, y) = [\ln(x^2 + y^2)]/\sqrt{x^2 + y^2}$ over the region $1 \leq x^2 + y^2 \leq e$.

2. Converting to a polar integral : Integrate
   $f(x, y) = [\ln(x^2 + y^2)]/(x^2 + y^2)$ over the region $1 \leq x^2 + y^2 \leq e^2$.

3. Volume of noncircular right cylinder : The region that lies inside
   the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ is the base of a
   solid right cylinder. The top of the cylinder lies in the plane $z = x$.
   Find the cylinder's volume.

4. Volume of noncircular right cylinder : The region enclosed by the
   lemniscate $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top
   is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

Solution for the Exercise 14

1. $\int_0^{2x} \int_1^{\sqrt{e}} \left(\frac{\ln\ r^2}{t} \right) r\ dr\ d\theta = \int_0^{2x} \int_1^{\sqrt{e}} 2\ln\ r\ dr\ d\theta = 2 \int_0^{2x} [r\ln\ r - r]_1^{e\sqrt{e}} d\theta =$

   $2 \int_0^{2x} \sqrt{e}[(\frac{1}{2} - 1) + 1]d\theta = 2\pi(2 - \sqrt{e})$

2. $\int_0^{2x} \int_1^{e} \left(\frac{\ln\ r^2}{r} \right) dr\ d\theta = \int_0^{2x} \int_1^{e} \left(\frac{2\ln r}{r} \right) dr\ d\theta = \int_0^{2x} [(\ln\ r)^2]_1^{e} d\theta = \int_0^{2x} d\theta = 2\pi$

3. $V = 2 - \int_0^{x/2} \int_1^{1+\cos\theta} r^2 \cos\theta dr\ d\theta = \frac{2}{3} \int_0^{x/2} (3\ cos^2\theta + 3\ cos^3\theta + \cos^4\theta)d\theta =$

   $\frac{2}{3} \left[\frac{15\theta}{8} + \sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{\sin 4\theta}{32} \right]_0^{x/2} = \frac{4}{3} + \frac{5\pi}{8}$

4. $V = 4 \int_0^{x/4} \int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2 - r^2} dr\ d\theta = -\frac{4}{3} \int_0^{x/4} [(2 - 2\cos 2\theta)^{3/2} - 2^{3/2}]d\theta = \frac{2\pi\sqrt{2}}{3} -$

   $\frac{32}{3} \int_0^{x/4} (1 - \cos^2\theta)\sin\ \theta\ d\theta = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3} \left[\frac{\cos^3\theta}{3} - \cos\theta \right]_0^{x/4} = \frac{6\pi\sqrt{2} + 40\sqrt{2} - 64}{9}$

Theorem 1 (Separation of Variables for Iterated Integrals).

Let $g(x)$ be a continuous function on the interval $[a, b]$ on the x-axis and $h(y)$ be a continuous function on the integral $[c, d]$ on the y-axis. Then $f(x, y) = g(x)h(y)$ is a continuous function on the rectangle $D = [a, b] \times [c, d]$ and

$$\iint_D f(x, y) \ dA = \Big(\int_a^b g(x) \ dx \Big) \Big(\int_c^d h(y) \ dy \Big).$$

Exercise 15.

(a) *The usual way to evaluate the improper integral $I = \int_0^\infty e^{-x^2} \ dx$ is first to calculate its square:*

$$I^2 = \Big(\int_0^\infty e^{-x^2} \ dx \Big) \Big(\int_0^\infty e^{-y^2} \ dy \Big) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \ dx \ dy.$$

Evaluate the last integral using polar coordinates and solve the resulting equation for I.

(b) *Evaluate*

$$\lim_{x \to \infty} erf(x) = \lim_{x \to \infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} \ dt.$$

Solution for the Exercise 15

(a)

$$
\begin{aligned}
I^2 &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx \ dy \\
&= \int_0^{x/2} \int_0^\infty \left(e^{-r^2} \right) r \ dr \ d\theta \\
&= \int_0^{x/2} \left[\lim_{b \to \infty} \int_0^\infty re^{-r^2} dr \right] d\theta \\
&= -\frac{1}{2} \int_0^{x/2} \lim_{b \to \infty} \left(e^{-b^2} - 1 \right) d\theta \\
&= \frac{1}{2} \int_0^{x/2} d\theta = \frac{\pi}{4}.
\end{aligned}
$$

Hence $I = \frac{\sqrt{\pi}}{2}$.

(b) $\displaystyle \lim_{x \to \infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} dt = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = \left(\frac{2}{\sqrt{\pi}} \right) \left(\frac{\sqrt{\pi}}{2} \right) = 1$, from part (a).

Converting to polar integrals

Exercise 16.

Converting to a polar integral : *Evaluate the integral*

$$\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy.$$

Solution for the Exercise 16

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dx \ dy \ = \ \int_0^{\pi/2} \int_0^\infty \frac{r}{(1+r^2)^2} dr \ d\theta$$

$$= \ \frac{\pi}{2} \lim_{b\to\infty} \int_0^b \frac{r}{(1+r^2)^2} dr$$

$$= \ \frac{\pi}{4} \lim_{b\to\infty} \Big[-\frac{1}{1+r^2} \Big]_0^b$$

$$= \ \frac{\pi}{4} \lim_{b\to\infty} \left(1 - \frac{1}{1+b^2} \right) = \frac{\pi}{4}$$

Exercises

Exercise 17.

1. Existence : *Integrate the function $f(x, y) = 1/(1 - x^2 - y^2)$ over the disk $x^2 + y^2 \leq 3/4$. Does the integral of $f(x, y)$ over the disk $x^2 + y^2 \leq 1$ exist? Give reasons for your answer.*

2. Area : *Suppose that the area of a region in the polar coordinate plane is*

$$A = \int_{\pi/4}^{3\pi/4} \int_{\csc\theta}^{2\sin\theta} r \, dr \, d\theta.$$

Sketch the region and find its area.

Solution for (1.) in Exercise 17

Over the disk $x^2 + y^2 \leq \frac{3}{4}$,

$$
\begin{aligned}
\iint_R \frac{1}{1 - x^2 - y^2} dA &= \int_0^{2x} \int_0^{\sqrt{3/2}} \frac{r}{1 - r^2} dr \, d\theta \\
&= \int_0^{2x} \left[-\frac{1}{2} \ln (1 - r^2) \right]_0^{\sqrt{3}/2} d\theta \\
&= \int_0^{2x} \left(-\frac{1}{2} \ln \frac{1}{4} \right) d\theta \\
&= (\ln \ 2) \int_0^{2x} d\theta = \pi \ln 4.
\end{aligned}
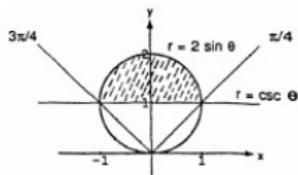$$

Solution for (1.) in Exercise 17 (contd...)

Over the disk $x^2 + y^2 \leq 1$,

$$
\begin{aligned}
\iint_R \frac{1}{1 - x^2 - y^2} dA &= \int_0^{2x} \int_0^1 \frac{r}{1 - r^2} dr \ d\theta \\
&= \int_0^{2x} \left[\lim_{a \to 1} \int_0^a \frac{r}{1 - r^2} dr \right] d\theta \\
&= \int_0^{2x} \lim_{a \to 1} \left[-\frac{1}{2} \ln(1 - a^2) \right] d\theta \\
&= 2\pi \cdot \lim_{a \to 1} \left[-\frac{1}{2} \ln(1 - a^2) \right] = 2\pi . \infty
\end{aligned}
$$

so the integral does not exist over $x^2 + y^2 \leq 1$.

Solution for (2.) in Exercise 17

$$
\begin{aligned}
A &= \int_{x/4}^{3x/4} \int_{\csc\theta}^{2\sin\theta} r \; dr \; d\theta \\
&= \frac{1}{2} \int_{x/4}^{3x/4} (4\sin^2\theta - \csc^2\theta) d\theta \\
&= \frac{1}{2} [2\theta - \sin 2\theta + \cot\theta]_{x/4}^{3x/4} = \frac{\pi}{2}
\end{aligned}
$$

Exercises

Exercise 18.

1. Area formula in polar coordinates : *Use the double integral in polar coordinates to derive the formula*

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

   for the area of the fan-shaped region between the origin and polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$.

2. Average distance to a given point inside a disk : *Let P_0 be a point inside a circle of radius a and let h denote the distance from P_0 to the center of the circle. Let d denote the distance from an arbitrary point P at P_0. Find the average value of d^2 over the region enclosed by the circle. (Hint: Simplify your work by placing the center of the circle at the origin and P_0 on the x-axis.)*

Solution for the Exercise 18

1. The area in polar coordinates is given by
$$A = \int_\alpha^\beta \int_0^{f[\theta]} r \; dr \; d\theta = \int_\alpha^\beta \left[\frac{r^2}{2}\right]_\alpha^\beta d\theta = \frac{1}{2}\int_\alpha^\beta f^2(\theta)d\theta = \int_\alpha^\beta \frac{1}{2}r^2 d\theta, \text{ where } r = f(\theta).$$

2.

$$
\begin{aligned}
average &= \frac{1}{\pi a^2}\int_0^{2x}\int_0^a [(r\cos\theta - h)^2 + r^2 \sin^2\theta] r \; dr \; d\theta \\
&= \frac{1}{\pi a^2}\int_0^{2x}\int_0^a (r^3 - 2r^2 h\cos\theta + rh^2) dr \; d\theta \\
&= \frac{1}{\pi a^2}\int_0^{2x}\left(\frac{a^4}{4} - \frac{2a^3 h\cos\theta}{3} + \frac{a^2 h^2}{2}\right) d\theta \\
&= \frac{1}{\pi}\int_0^{2x}\left(\frac{a^2}{4} - \frac{2ah\cos\theta}{3} + \frac{h^2}{2}\right) d\theta \\
&= \frac{1}{\pi}\left[\frac{a^2\theta}{4} - \frac{2ah\sin\theta}{3} + \frac{h^2\theta}{2}\right]_0^{2x} \\
&= \frac{1}{2}(a^2 + 2h^2)
\end{aligned}
$$

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